Rabi oscillations at exceptional points in microwave billiards

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We experimentally investigated the decay behavior with time t of resonances near and at exceptional points, where two complex eigenvalues and also the associated eigenfunctions coalesce. The measurements were performed with a dissipative microwave billiard, whose shape depends on two parameters. The t^2 dependence predicted at the exceptional point on the basis of a two-state matrix model could be verified. Outside the exceptional point the predicted Rabi oscillations, also called quantum echoes in this context, were detected.

PACS number(s): 05.45.Mt, 41.20.Jb, 03.65.Nk, 03.65.Vf

In quantum mechanics many dynamical processes are dominated by (avoided) level crossings. A crossing of two eigenvalues requires the variation of at least two parameters. It has been known for many years that near a crossing the two energy surfaces form two sheets of a double cone [1,2]. The apex of the double cone is associated with a singularity and called a diabolic point (DP) [3,4]. A DP occurs in Hermitian Hamiltonians. Phenomena related to a DP, as, e.g., geometric phases, have been studied theoretically in various generalizations of Berry's original paper (see, e.g., [5,6] and references therein) and experimentally, e.g., in [7] with a microwave billiard. For non-Hermitian Hamiltonians, as those used for the theoretical description of dissipative systems, a topologically different singularity may appear: an exceptional point (EP) [8]—there not only the eigenvalues but also the associated eigenstates coalesce [9]. Thus EPs are not only singularities of the spectrum but also of the eigenstates. They have been observed in laser induced ionizations of atoms [10], crystals of light [11], electronic circuits [12], the propagation of light in dissipative media [13,14], and in microwave billiards [15–17] and also appear in many theoretical models: e.g., in that used for the decay of superdeformed nuclei [18], phase transitions and avoided level crossings [19,20], geomagnetic polarity reversal [21], tunneling between quantum dots [22], and in the context with the crossing of two Coulomb blockade resonances [23].

Exceptional points give rise to interesting phenomena such as level crossings and geometric phases [4,17,24]. In this Brief Report we present interesting experimental results on the time decay of resonances in the vicinity of and at an EP. Close to an EP, the time spectrum exhibits—besides the decay of the isolated resonances—oscillations with a fixed frequency; these are called quantum echoes [25]. Quantum echoes occur due to the transfer of energy between the two nearly degenerate resonances. At the EP their vanishing and a quadratic time dependence of the resonance amplitude was predicted. This time dependence is a characteristic property of EPs with exactly two coinciding eigenvalues. In general more than two eigenvalues may coincide at an EP, how-

ever, the coincidence of two eigenvalues is the most probable.

For wavelengths longer than twice the height of the microwave billiard the scalar Helmholtz equation of the electric field strength in a cylindric microwave billiard is equivalent to the Schrödinger equation for the wave functions in a quantum billiard of corresponding shape (see, e.g., [26,27]). Thus aside from their intrinsic interest, flat microwave billiards yield a possibility to gain experimental insight into properties of the eigenvalues and eigenfunctions of two-dimensional quantum billiards. We used a coupled pair of dissipative, cylindric, and flat microwave billiards manufactured of copper to mimic the two-level system required for an EP to occur [15–17]; a sketch is shown in Fig. 1. It is obtained by dividing a circular microwave billiard with a barrier made from copper into two approximately equal parts differing in their areas by about 5%. Due to this slight difference in size the twofold degeneracy of the eigenfrequencies of the circular billiard is lifted, that is the
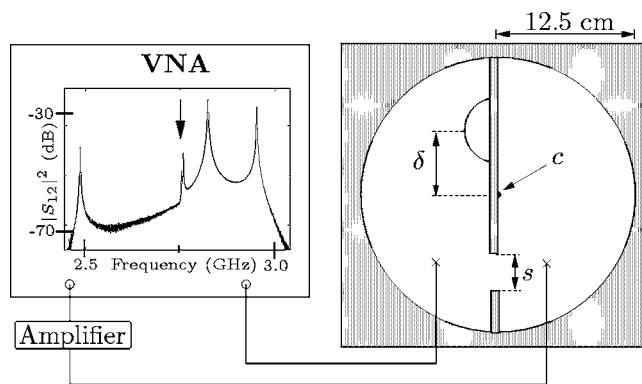

FIG. 1. Sketch of the experimental setup: In the microwave billiard (right part) the parameter s refers to the length of the slit and δ to the position of the semicircular Teflon® disc, measured from the billiard center c. The antenna positions are shown as crosses. The vectorial network analyzer (left part) measures the ratio of the power of the signals received at the left antenna 1 and emitted at the right antenna 2, $|S_{12}|^2$, and their relative phase. Shown is a part of a typical transmission spectrum measured in dB. The arrow points at two nearly degenerate resonances which for a certain choice of the parameters s and δ coalesce at the exceptional point at 2.757 ± 0.001 GHz.

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eigenfrequencies are split into two nearly degenerate ones. Their crossing behavior has been analyzed in [15–17] with a slightly modified experimental setup.

In order to control the coalescence of two states, we must be able to control their frequencies and widths. We achieved this by tuning two parameters, the length of slit s between the two parts of the microwave billiard and the position δ of a semicircular Teflon® disc (see Fig. 1). The parameter δ mainly affects the resonance frequencies of the billiard part which contains the Teflon® disc, whereas the parameter s controls the coupling between the eigenmodes of the two billiard parts. Coupling zero is achieved by closing the slit. Then, due to the Dirichlet boundary condition, the electric field strength vanishes alongside the barrier. When opening the slit, those eigenmodes of the resonators are affected, which have a nonvanishing field intensity near the slit. Depending on their derivatives in normal direction to the slit, they penetrate into the other cavity and couple to its modes.

To measure the transmission spectrum of the billiard system the vectorial network analyzer (VNA) "HP-8510C" produces an rf signal, which is piped through a 30-dB amplifier and is forwarded to a dipole antenna 0.5 mm in diameter and 2 mm in length. If the signal frequency is tuned to a resonance frequency of the billiard, the dipole antenna excites the corresponding eigenmode such that power is radiated into the microwave billiard—otherwise the power is reflected at the antenna. The power is coupled out via a second antenna and lead back into the VNA, which compares the produced and the received signal in amplitude and phase. The squared eigenfunctions are measured with the perturbation body method. This method is based on the Maier-Slater theorem [28], which states that the amount of the frequency shift of a resonance depends on the difference of the square of the electric field strength and of the magnetic induction at the position of the perturbing body. Hence the intensity distribution of the electric field strength is obtained by moving the perturbation body through the microwave billiard on a grid with spacing 5 mm and comparing the perturbed and unperturbed resonance frequency. As perturbation we use a cylindrical magnetic rubber about 2.2 mm in diameter and 3.5 mm in height. Magnetic rubber has the advantageous property that it suppresses the contribution of the magnetic induction to the frequency shift and still enables us to change its position inside the closed microwave billiard with a guide magnet.

Quantum echoes are time recurrent signals. They have been seen, e.g., in open microwave billiards [25] but they also appear in the gravity induced interference effect [29], the Aharonov-Bohm effect [30], as well as in NH₃ MASERs (Microwave Amplification by Stimulated Emission with Radiation) described, e.g., in [31]. In the latter they are better known as a beating with a frequency called Rabi frequency. The decay behavior with time of two resonance states which are near an EP, i.e., the superposition of the energy transfer between the two resonances and that to the exterior, is analyzed with a two-level Hamiltonian which was used by Stafford and Barrett [18] in a different context. The theoretical results presented there may be mapped one to one onto our problem. This signifies that the experimental results presented here are of general interest in different fields of physics dealing with EPs.

The electric field strengths $\vec{\mathbb{E}}_j(t) = E_j(t)\vec{e}_z$ with $j = 1, 2$ of the eigenmodes in the two parts of the microwave billiard can be described as a pair of coupled damped harmonic oscillators,

$$\left(\frac{\partial^2}{\partial t^2} + \gamma_1 \frac{\partial}{\partial t} + \omega_1^2\right)E_1(t) = F_1(t) - \sigma E_2(t),$$

$$\left(\frac{\partial^2}{\partial t^2} + \gamma_2 \frac{\partial}{\partial t} + \omega_2^2\right)E_2(t) = F_2(t) - \sigma E_1(t), \quad (1)$$

with angular frequencies ω_1 and ω_2, respectively, decay rates γ_1 and γ_2, and a coupling strength σ. The driving terms $F_j(t)$ with $j = 1, 2$ describe the coupling of the antennae to the eigenmodes. The coupled differential equations (1) can be solved with the Green's function method. The inverse of Green's function is given by $\mathbb{G}^{-1}(\omega) = \omega^2 \mathbb{1} - \mathbb{H}(\omega)$ with the non-Hermitian two-level Hamiltonian [16,17]

$$\mathbb{H}(\omega) = \begin{pmatrix} \omega_1^2 - i\omega\gamma_1 & \sigma \\ \sigma & \omega_2^2 - i\omega\gamma_2 \end{pmatrix}. \quad (2)$$

While the real parts of the diagonal elements (ω_1^2 and ω_2^2) are proportional to the square of the resonance frequencies of the uncoupled billiard parts, the imaginary parts ($\Gamma_1 \equiv \omega\gamma_1$ and $\Gamma_2 \equiv \omega\gamma_2$) describe their resonance widths. The Hamiltonian is symmetric since the system is time reversal invariant and it is non-Hermitian due to the antennae and the absorption in the billiard walls and the Teflon® disc. We denote the eigenvalues of Eq. (2) by $\mathcal{E}_\pm := \bar{\omega}^2 - i\bar{\Gamma} \pm R$ with $R := 1/2\sqrt{[\Delta\omega^2 - i\Delta\Gamma]^2 + 4\sigma^2}$, $\bar{\omega} := \sqrt{(\omega_1^2 + \omega_2^2)/2}$, $\bar{\Gamma} := (\Gamma_1 + \Gamma_2)/2$, $\Delta\omega^2 := \omega_2^2 - \omega_1^2$, and $\Delta\Gamma := \Gamma_2 - \Gamma_1$. Although in the experiment the physical quantities ω_1, ω_2, Γ_1, and Γ_2 are complicated functions of the Teflon® disc position δ and the opening length s (see Fig. 1), these two parameters are sufficient to achieve coalescence of the two eigenvalues of the two-level Hamiltonian, i.e., the vanishing of the parameter R. At the EP $\Delta\omega^2$ vanishes and $\Delta\Gamma^2 = 4\sigma^2$. The parameter setting is denoted by $s = s^{\text{EP}}$ and $\delta = \delta^{\text{EP}}$. As we are interested in the decay behavior with time of the two-level system we need to compute the Fourier transformation from the frequency to the time domain of $\mathbb{G}(\omega)$. The resulting expression is quite complicated. However, in the parameter range of the experiment a simple and accurate analytic expression is obtained with the following approximation: in the vicinity of $\omega \simeq \omega_1, \omega_2$ the terms $\omega\gamma_1$ and $\omega\gamma_2$ are small in comparison with ω^2. Therefore ω in Eq. (2) is replaced by the average value of the angular frequencies $\bar{\omega}$. This approximation takes into account that the major contribution comes from the poles of the Green function. With

$$\Omega := \frac{R}{2\bar{\omega}}, \quad f := \bar{\omega} - \frac{i}{2}\bar{\gamma}, \quad \bar{\gamma} := \frac{\gamma_1 + \gamma_2}{2} \quad (3a)$$

the Fourier transform $\widetilde{\mathbb{G}}_{12}(t)$, which describes the energy transfer from one resonance to another, is given by [18,22]

$$|\widetilde{G}_{12}(t)|^2 \approx \frac{\sigma^2}{\bar{\omega}^2} \left| \frac{e^{-ift}}{\sqrt{\mathcal{E}_+ \mathcal{E}_-}} \left(\cos(\Omega t) + if \frac{\sin(\Omega t)}{\Omega} \right) \right|^2. \quad \text{(3b)}$$

Hence $\widetilde{G}_{12}(t)$ decays exponentially with a decay constant of $\bar{\gamma}/2$ and oscillates with a high, not resolvable angular frequency $\bar{\omega}$. The physical value $\Omega/2\pi$ denotes the much lower $(R \ll \bar{\omega})$ *echo frequency*. It corresponds to the Rabi frequency first observed in NMR and optical pumping [32] and also well known in quantum optics [33], nuclear physics [18], and quantum computing [34].

As was pointed out already in [18,22] the imaginary part of the echo frequency $\Omega/2\pi$ adds to the decay, while the real part describes the oscillation between the two resonances. As a consequence, the quantum echoes vanish if Ω is purely imaginary. This happens *for all* subcritical couplings, that is for slit openings $s < s^{\text{EP}}$ at the critical Teflon® disc position $\delta = \delta^{\text{EP}}$. For overcritical couplings, i.e., for $s > s^{\text{EP}}$, the quantum echoes persist for all Teflon® disc positions, while exactly at an EP both the real and imaginary part of the echo frequency $\Omega/2\pi$ vanish. Then, with Eq. (3b) a quadratic time dependency of the echo amplitude

$$\lim_{(s,\delta) \to \text{EP}} |\widetilde{G}_{12}(t)|^2 \approx \frac{\sigma^2}{\mathcal{E}_+ \mathcal{E}_-} t^2 e^{-\bar{\gamma} t} \quad \text{(4)}$$

is obtained. This dependency can also be verified by an exact calculation based on Eq. (2) evaluated at the EP. It is consistent with the result given in [35].

The EPs were localized in the parameter plane $\{s, \delta\}$ by varying the position of the Teflon® disc and the slit opening and measuring crossings and avoided crossings of the frequencies and widths of the resonances which coincide at the EP [36]. In parallel we performed measurements of the nodal domains of the eigenfunction. For a more detailed description see, for example, [17]. Since the time behavior is extremely sensitive in the vicinity of an EP, the Teflon® disc position and the opening of the slit were changed in steps of 0.5 mm. We were able to localize two EPs below 3 GHz, the first at 2.757 ± 0.001 GHz and the second at 2.806 ± 0.001 GHz. In this frequency range the pairs of resonances which coalesce at the EP can be treated as isolated. Hence the two-level model is applicable. Since both EPs exhibit the same decay behavior with time, we only show the results for the first.

The decay of the resonances with time was deduced from the transmission spectra by a fast Fourier transformation. For its computation a narrow frequency range of about 0.3 GHz was chosen around the doublet and a Hamming window function was used. We checked that the choice of the latter does not affect the results. The square of this transformation and a fit of Eq. (3b) to these data are shown in Fig. 2 for the critical coupling, that is for $s = s^{\text{EP}}$ and a subcritical Teflon® disc position $\delta < \delta^{\text{EP}}$. The model [Eq. (3b)] describes the measured signal very well starting from a time larger than 30 ± 5 ns, where a peak indicates the time the signal needs to travel through the coaxial cables connecting the VNA with the antenna, up to times, where the noise level is reached (at about -65 dB).
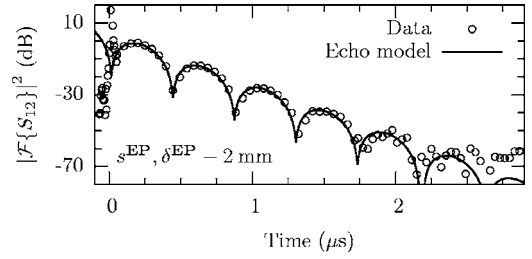

FIG. 2. The model [Eq. (3b)] describes the time spectrum very well. This is shown here for the exceptional point at 2.757 ± 0.001 GHz for the parameter setting $s = s^{\text{EP}} \approx 48$ mm, $\delta = \delta^{\text{EP}} - 2$ mm ≈ 21.5 mm. The echo frequency is $\Omega/2\pi \approx 1.1 \pm 0.2$ MHz.

The echo frequency equals $\Omega/2\pi = 1.1 \pm 0.2$ MHz, which approximately corresponds to the spacing between the two eigenfrequencies.

The time decay of the resonances for $s = s^{\text{EP}}$ and $\delta = \delta^{\text{EP}}$ is shown in Fig. 3 together with the theoretical prediction [Eq. (4)]. The quantum echoes disappear as they are an interference effect of two eigenmodes. Moreover, the coalescence of the two eigenmodes has an influence on the time decay behavior. An isolated resonance decays simply exponentially, whereas at an EP the time dependence of the echo amplitude is quadratic. Hence the time decay behavior of the resonances provides information on the nature of a degeneracy, and, consequently, the location of EPs in the parameter plane. In order to check how close we are to the EP we performed an additional fit of the function in Eq. (4) to the experimental data, however, this time with a t^α instead of a t^2 dependence of the echo amplitude, yielding $\alpha = 1.91 \pm 0.04$. This small deviation from the t^2 dependence and that observed in Fig. 3 for times larger than $t \approx 2$ μs are due to the fact that the EP is not hit exactly. It expresses how sensitive this experiment is to tiny changes in the parameters. If the real parts of the two eigenvalues of the Hamiltonian [Eq. (2)] coincide, R and therefore Ω are purely imaginary for subcritical coupling, where $s < s^{\text{EP}}$, and we expect for the time dependence a sum of exponentials. Equation (3b) again provides a very good description of the measured time dependence. However, we were not able to verify that at the critical Teflon position $\delta = \delta^{\text{EP}}$ the quantum echoes vanish for subcritical couplings. It seems that the resonance frequencies ω_1 and ω_2 of the two uncoupled resonators are only approxi-
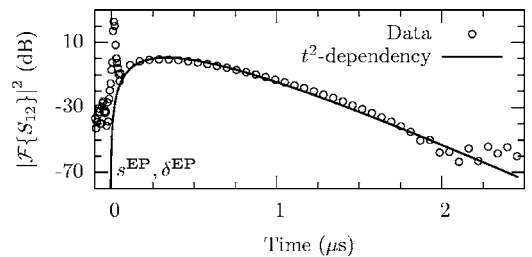

FIG. 3. At the EP the quantum echoes disappear and the predicted quadratic time decay of the echo amplitude could be verified. Shown is the result for the EP at 2.757 ± 0.001 GHz together with the theoretical prediction [Eq. (4)].

mately degenerate, such that the real part of Ω in Eq. (3b) is very small but nonvanishing.

In the present work we experimentally studied quantum echoes in the vicinity of and at two exceptional points (EPs). Although the system is very sensitive to small deviations of the critical parameters, we were able to control the system sufficiently well to verify the predicted t^2 dependence of the echo amplitude at an EP. We showed that a 2×2 model describes the quantum echoes very well. In addition, the determination of the time dependence of the transfer of energy between two closely lying resonances provides a straightforward method for the location of exceptional points in the parameter plane.

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